JOURNAL OF NUMBER THEORY 3, 111–114 (1971)

The Frattini Subgroups of Linear Noetherian Groups

SURINDER K. SEHGAL AND HANS ZASSENHAUS

Department of Mathematics, Ohio State University, Columbus, Ohio 43210

Communicated October 11, 1969

If G is a linear Noetherian group, then (a) $\Phi(G)$, the Frattini subgroup of G is nilpotent; (b) if $G/\Phi(G)$ is nilpotent, then G is nilpotent.

The Frattini subgroup $\Phi(G)$ of a group G can be defined as (i) the intersection of G and all maximal subgroups of G or (ii) the union of all elements g of G that can be omitted from any set of generators for G, i.e., of all g for which $\{g, M\} = G \Rightarrow M = G$. The equivalence of (i) and (ii) was proved for finite groups by Frattini and for arbitrary groups by B. H. Neuman [5]. It is well-known that the Frattini subgroup of a finite group is nilpotent and also if $G/\Phi(G)$ is nilpotent then so also is G. K. Hirsch [3] and N. Itô [4] have shown that $\Phi(G)$ is nilpotent whenever G is polycyclic (equivalently G is noetherian and solvable). P. Hall [2] has shown that $\Phi(G)$ is not nilpotent in general. R. Baer [1] has conjectured that every noetherian group contains a normal, polycyclic subgroup of finite index. H. Zassenhaus [6] has shown that Baer's conjecture is true in the case of linear noetherian groups. In this paper, we will show that if G is a linear noetherian Group, then (1) $\Phi(G)$ is nilpotent; (2) if $G/\Phi(G)$ is nilpotent, then G is nilpotent. In particular, the results are true for polycyclic groups. V. P. Platonov [8] has proved using a slight generalization of Malcev's approximation theorem [9] that if G is finitely generated linear group, then $\Phi(G)$ is nilpotent. The proofs of our theorems are rather elementary.

Notation

A. If G is a group with a normal chain such that every factor either is finite or infinite cyclic, then the number of infinite cyclic factors is called the rank of G. It is unique by the Jordan–Hölder–Schreier theorem [7].

B. $N \lhd G$ stands for N is a normal subgroup of G. $Z(G)$ is the center of G. $G_i = G_1, G_2, G_3, \ldots$ is the descending central series of G. $C_G(A)$ is the centralizer of A in G.

The following are well-known results, which will be used in the paper.

I. If G is nilpotent and N is a normal subgroup of G, then N intersects the center of G nontrivially.

II. If G is nilpotent and rank $G \geqslant 1$, then the rank of the center of G is $\geqslant 1$.

III. If G is a nonnilpotent polycyclic group, then G contains a characteristic subgroup N of finite index such that G/N is not nilpotent [3].

THEOREM 1. *If G is finite over polycyclic,*[1] *then $\Phi(G)$ is nilpotent.*

Proof. Suppose the result is not true. Let G be a group of smallest possible rank for which the result is false. Let S be the set consisting of $\{N \mid N \lhd G, N \subseteq \Phi(G)$ and such that $\Phi(G)/N$ is not nilpotent$\}$. Now S is not an empty set. Since G is noetherian, S has a maximal element say M_0. So that in G/M_0, we have $\Phi(G)/M_0 = \Phi(G/M_0)$ is not nilpotent. We can replace G by G/M_0 and say that $\Phi(G)$ is not nilpotent, but the image of $\Phi(G)$ under every nontrivial homomorphism of G is nilpotent and G is of minimal rank among the groups for which Theorem 1 is not true. Now $\Phi(G)$ contains a polycyclic subgroup P of finite index such that P is normal in G. In case P is finite, then $\Phi(G)$ is also finite. Let S_p be a sylow p-subgroup of $\Phi(G)$. Then $N_G(S_p) \cdot \Phi(G) = G \Rightarrow N_G(S_p) = G \Rightarrow S_p \lhd G \Rightarrow \Phi(G)$ is nilpotent which is a contradiction. So we can assume P is not finite. We claim that P is nilpotent. By Ref. [3], P contains a characteristic subgroup H such that P/H is finite and not nilpotent. But by our assumptions $P/H \subseteq \Phi(G/H) = \Phi(G)/H$ is nilpotent, this is a contradiction. Now $Z(P)$ is a characteristic subgroup of P, and $P \lhd G$, which implies $Z(P) \lhd G$.

Subcase (i). $Z(P)$ has a torsion element $\neq 1$. Then $Z(P)$ contains a minimal characteristic subgroup N of finite order. Now $\Phi(G)/N$ is nilpotent and N is an elementary abelian p-group for some prime p. Let $\bar{Z}$ be the inverse image of the center of $\Phi(G/N)$ under the natural homomorphism: $G \to G/N$. Let $x \in \bar{Z} \cap C_G(N)$ and $|x| = \infty$ [such an element exists since $G/C_G(N)$ is of finite order].

Let y be any element of $\Phi(G)$, then $[y, x] \equiv 1 \pmod{N}$, i.e., $y^{-1}xy = xn$

[1] A group is said to be "finite over polycyclic" if it contains a polycyclic subgroup of finite index. Since the core of a subgroup of finite index itself has finite index and since any subgroup of a polycyclic group is polycyclic, the meaning of "finite over polycyclic" may be restricted to "existence of a polycyclic normal subgroup of finite index." According to Ref. [6] this concept is equivalent to "linear noetherian." This remark explains why instead of the "linear noetherian" of the title we have usually employed the expression "finite over polycyclic." No use was made anywhere of the linearity of the groups under consideration.

with n in N, or we have $y^{-1}x^py = x^p$. Thus, we see that $x^p \in Z(\Phi(G))$. So the center of $\Phi(G)$ has rank $\geqslant 1$. Hence $G/Z(\Phi(G))$ has rank less than rank of G. By assumption its Frattini subgroup is nilpotent, i.e., $\Phi(G)/Z(\Phi(G))$ is nilpotent. Hence $\Phi(G)$ is nilpotent, a contradiction.

Subcase (ii). $Z(P)$ is torsion free, abelian, say of rank r. Let $N = Z(P)$. Now $\Phi(G)/N^q$ is nilpotent, where q is a prime. Since $N : N^q = q^r$; it follows from $\Phi_k(G) \subseteq N$ that $\Phi_{k+r}(G) \subseteq N^q$ for infinitely many primes q. Therefore $\Phi_{k+r}(G) = 1$ and hence $\Phi(G)$ is nilpotent.

THEOREM 2. *If G is finite over polycyclic and if N is a normal subgroup of G, contained in the Frattini subgroup of G, such that G/N is nilpotent then G is nilpotent.*

Proof. Suppose the result is not true. Suppose G is a group of smallest possible rank for which the result is false. Let S be the set

$$\{M/M \lhd G, M \subseteq \Phi(G) \quad \text{and} \quad G/M \text{ not nilpotent}\}.$$

Then S is not empty. Since G is noetherian, S has a maximal element say M_0. We can replace G by G/M_0 and assume that G/N is nilpotent for all normal subgroups N of G for which $1 \neq N \subseteq \Phi(G)$, G is not nilpotent and G is of smallest possible rank. Then there is also a normal abelian subgroup N of G such that $1 \neq N \subseteq \Phi(G)$. Hence, there is a natural number k for which $G_k \subseteq N$.

Case (i).

N has a torsion element. Let M be a minimal normal subgroup of G of finite order and $M \subseteq N \subseteq \Phi(G)$ and G/M is nilpotent and M is a elementary abelian p-group for some prime p. Let $\bar{Z}$ be the inverse image of the center of G/M under the natural homomorphism: $G \to G/M$.

Let $x \in \bar{Z} \cap C_G(M)$ and $|x| = \infty$. (Such an x exists since rank $C_G(M) =$ rank G). Let y be any element of G. Then $y^{-1}xy = xm$ with m in M, or $y^{-1}x^py = x^p$, or $x^p \in Z(G)$. Replace x^p by x. We have $G/\langle x\rangle$ is a group of smaller rank than rank (G). Therefore $M\langle x\rangle/\langle x\rangle \not\subseteq \Phi(G/\langle x\rangle)$, i.e., $\exists$ a maximal subgroup $L/\langle x\rangle$ such that $L \neq G$ and $L \not\supseteq M\langle x\rangle$, which cannot happen.

Case (ii).

N is torsion free. Since it is polycyclic abelian, it is a free abelian of finite rank r. Now G/N^q has rank less than rank of G. Therefore G/N^q is nilpotent. Since $N : N^q = q^r$, it follows from $G_k \subseteq N$ that $G_{k+r} \subseteq N^q$. Since G_{k+r} is contained in N^q for infinitely many primes q, we conclude that $G_{k+r} = 1$. Therefore, G is nilpotent.

References

1. R. Baer, Noethersche Gruppen I, *Math. Z.* **66** (1956), 269–88; Part II, *Math. Ann.* **165** (1966), 163–80.
2. P. Hall, *Proc. London Math. Soc.* **11** (1961), 327–52.
3. K. A. Hirsch, *J. London Math. Soc.* **29** (1954), 250–51.
4. N. Itô, *Proc. Japan Acad.* **29** (1953), 149–50.
5. B. H. Neuman, *J. London Math. Soc.* **12** (1937), 120–27.
6. H. Zassenhaus, On Linear Noetherian Groups, *J. Number Theory* **1** (1969), 70–89.
7. H. Zassenhaus, "The Theory of Groups," 2nd. ed., Chelsea, New York, 1958.
8. V. P. Platonov, The Frattini subgroup of Linear Groups and finite approximability, *Soviet Math. Dokl.* **7** (1966), 1557–60.
9. A. I. Malcev, *Mat. Sb.* **50** (1940), 405; *Amer. Math. Soc. Transl.* (2) **45** (1965), 1. MR 2, 216.